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Existence and multiplicity of symmetric solutions for semilinear elliptic equations with singular potentials and critical Hardy–Sobolev exponents ☆

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ABSTRACT

This paper deals with the singular semilinear elliptic problem

$$-\mathrm{div}(|x|^{-2a}\nabla u) = \mu\frac{u}{|x|^{2(1+a)}} + Q(x)\frac{|u|^{p-2}u}{|x|^{bp}} + \sigma h(x,u)$$
$$\text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega,$$

where $\Omega \subset \mathbb{R}^N (N \geq 3)$ is a smooth bounded domain, $0 \in \Omega$ and Ω is G-symmetric with respect to a subgroup G of $O(\mathbb{N})$, $0 \leq a < \frac{N-2}{2}$, $\sigma \geq 0$, $0 \leq \mu < \overline{\mu}$ with $\overline{\mu} = (\frac{N-2-2a}{2})^2$, $a \leq b < a+1$, $p = p(a,b) = \frac{2N}{N-2(1+a-b)}$, $Q(x)$ is continuous and G-symmetric on $\overline{\Omega}$ and $h : \Omega \times \mathbb{R} \mapsto \mathbb{R}$ is a continuous nonlinearity of lower order satisfying some conditions. Based upon the symmetric criticality principle of Palais and variational methods, we prove several existence and multiplicity results of G-symmetric solutions under certain appropriate hypotheses on σ, Q and h.

1. Introduction

In recent years, people have paid much attention to the following singular elliptic problem

$$\begin{cases} -\Delta u = \mu\dfrac{u}{|x|^2} + |u|^{2^*-2}u + f(x,u), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a smooth domain (bounded or unbounded) in $\mathbb{R}^N (N \geq 3)$, $0 \in \Omega$, $0 \leq \mu < (\frac{N-2}{2})^2$, $2^* \triangleq \frac{2N}{N-2}$ is the critical Sobolev exponent, and $f(x,u)$ is the subcritical perturbation. This problem comes from the consideration of standing waves in the anisotropic Schrödinger equation. We also remark that Eq. (1.1) is related to applications in fluid mechanics and glaciology (see [1] for example). Due to this fact, many existence, nonexistence and multiplicity results for equations like (1.1) have been obtained with different hypotheses on the measurable function $f(x,u)$; see, for example [2–6] and the references therein. Moreover, for other results on this aspect, see [7] for boundary singularities, [8] for high-order nonlinearity, [9] for singular elliptic systems in $\mathbb{R}^2$, and [10] for non-autonomous Schrödinger–Poisson systems in $\mathbb{R}^3$.

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Recently, Deng and Jin [11] considered the existence of nontrivial solutions of the following singular semilinear elliptic problem

$$-\Delta u=\mu\frac{u}{|x|^2}+k(x)\frac{u^{2^*(s)-1}}{|x|^s}\quad \text{in } \mathbb{R}^N, \tag{1.2}$$

where $N>2$, $0\le s<2$, $0\le\mu<(\frac{N-2}{2})^2$, $2^*(s)=\frac{2(N-s)}{N-2}$, and k fulfills certain symmetry conditions with respect to a subgroup G of $O(\mathbb{N})$. By the variational arguments and analytic techniques, the authors proved the existence and multiplicity of G-symmetric solutions under certain conditions on k. Very recently, Deng and Huang [12] extended the results in [11] to quasilinear singular elliptic problems in a bounded G-symmetric domain. We also mention that when $\mu=s=0$ and the right-hand side term $u^{2^*(s)-1}$ is replaced by u^{r-1} ($1<r<\frac{2N}{N-2}$ or $r=\frac{2N}{N-2}$) in (1.2), the existence and multiplicity of G-symmetric solutions of (1.2) were obtained in [13–15]. Finally, when $G=O(\mathbb{N})$, we remark that Su and Wang [16] proved the existence of nontrivial radial solutions for a class of quasilinear singular equations such as (1.2) with radial potentials by establishing several new embedding theorems.

Motivated by [11,13,16], in this work we investigate the following singular semilinear elliptic problem of the type

$$\begin{cases}-\mathrm{div}(|x|^{-2a}\nabla u)=\mu\dfrac{u}{|x|^{2(1+a)}}+Q(x)\dfrac{|u|^{p-2}u}{|x|^{bp}}+\sigma h(x,u), & \text{in } \Omega,\\ u=0, & \text{on } \partial\Omega,\end{cases} \tag{1.3}$$

where $\Omega\subset\mathbb{R}^N$ ($N\ge 3$) is a bounded, G-symmetric domain (see Section 2 for details) with the smooth boundary $\partial\Omega$, $0\in\Omega$, $0\le a<\frac{N-2}{2}$, $\sigma\ge 0$, $0\le\mu<\overline{\mu}$, with $\overline{\mu}\triangleq(\frac{N-2-2a}{2})^2$, $a\le b<a+1$, $p=p(a,b)\triangleq\frac{2N}{N-2(1+a-b)}$ is the critical Hardy–Sobolev exponent and $p(a,a)=2^*=\frac{2N}{N-2}$ is the critical Sobolev exponent, $Q\in\mathcal{C}(\overline{\Omega})\cap L^\infty(\overline{\Omega})$ and $h\in\mathcal{C}(\Omega\times\mathbb{R},\mathbb{R})$ satisfy some conditions which will be specified later. Due to the nonlinear perturbation $\sigma h(x,u)$ and the singularities caused not only in the nonlinearities but also in the operator, compared with Eq. (1.2), the singular problem (1.3) becomes more complicated to deal with and we have to overcome more difficulties in the study of G-symmetric solutions. As far as we know, there are few results on the existence of G-symmetric solutions for (1.3) as $a\ne 0$, $b\ne 0$, $\mu\ne 0$ and $\sigma\ge 0$. It remains meaningful for us to investigate problem (1.3) deeply. Let $\overline{Q}>0$ be a constant. Note that, here, we will try to treat both the cases of $\sigma=0$, $Q(x)\not\equiv\overline{Q}$ and $\sigma>0$, $Q(x)\equiv\overline{Q}$.

This paper is organized as follows. In Section 2, we will establish the appropriate Sobolev space which is applicable to the study of problem (1.3), and will give the main results of this paper. In Section 3, we detail the proofs of some existence and multiplicity results for the case $\sigma=0$ and $Q(x)\not\equiv\overline{Q}$ in (1.3). In Section 4, we give the proofs of multiplicity results for the case $\sigma>0$ and $Q(x)\equiv\overline{Q}$ in (1.3). Our methods in this paper are mainly based upon the symmetric criticality principle of Palais (see [17]) and variational arguments.

2. Preliminaries and main results

Let $H_a^1(\Omega)$ denote the closure of $\mathcal{C}_0^\infty(\Omega)$ functions with respect to the norm $(\int_\Omega|x|^{-2a}|\nabla\cdot|^2dx)^{1/2}$. We recall that the well-known Caffarelli–Kohn–Nirenberg inequality (see [18]) asserts that for all $u\in H_a^1(\Omega)$, there is a constant $C_{a,b}>0$ such that

$$\left(\int_\Omega|x|^{-bp}|u|^pdx\right)^{2/p}\le C_{a,b}\int_\Omega|x|^{-2a}|\nabla u|^2dx, \tag{2.1}$$

where $-\infty<a<\frac{N-2}{2}$, $a\le b\le a+1$ and $p=\frac{2N}{N-2(1+a-b)}$. As $b=1+a$ and $p=2$, (2.1) becomes the following weighted Hardy inequality (see [19])

$$\int_\Omega|x|^{-2(1+a)}|u|^2dx\le\frac{1}{\overline{\mu}}\int_\Omega|x|^{-2a}|\nabla u|^2dx,\quad \forall u\in H_a^1(\Omega), \tag{2.2}$$

where $\overline{\mu}=(\frac{N-2-2a}{2})^2$. Now we employ the following norm in $H_a^1(\Omega)$,

$$\|u\|\triangleq\left(\int_\Omega\left(|x|^{-2a}|\nabla u|^2-\mu\frac{u^2}{|x|^{2(1+a)}}\right)dx\right)^{1/2},\quad 0\le\mu<\overline{\mu}.$$

By the weighted Hardy inequality (2.2) we easily see that the above norm is equivalent to the norm $(\int_\Omega|x|^{-2a}|\nabla u|^2dx)^{1/2}$.

Let $O(\mathbb{N})$ be the group of orthogonal linear transformations of $\mathbb{R}^N$ with natural action and let $G\subset O(\mathbb{N})$ be a subgroup. For $x\ne 0$ we denote the cardinality of $G_x=\{gx;\, g\in G\}$ by $|G_x|$ and set $|G|=\inf_{0\ne x\in\mathbb{R}^N}|G_x|$. Note that, here, $|G|$ may be $+\infty$. We call Ω a G-symmetric subset of $\mathbb{R}^N$, if $x\in\Omega$, then $gx\in\Omega$ for all $g\in G$. For any function $f:\mathbb{R}^N\to\mathbb{R}$, we call

$f(x)$ a G-symmetric function if for all $g \in G$ and $x \in \mathbb{R}^N$, $f(gx) = f(x)$ holds. In particular, if f is radially symmetric, then the corresponding group is $O(\mathbb{N})$ and $|G| = +\infty$. Other examples of G-symmetric functions can be found in [11].

For a bounded and G-symmetric domain $\Omega \subset \mathbb{R}^N$, $0 \in \Omega$, the natural functional space to study problem (1.3) is the Hilbert space $H^1_{a,G}(\Omega)$, which is the subspace of $H^1_a(\Omega)$ consisting of all G-symmetric functions. Now in this paper, we consider the following problems

$$(\mathscr{P}^Q_\sigma) \quad \begin{cases} -\mathrm{div}(|x|^{-2a}\nabla u) = \mu \dfrac{u}{|x|^{2(1+a)}} + Q(x)\dfrac{|u|^{p-2}u}{|x|^{bp}} + \sigma h(x,u), & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \text{ and } u \in H^1_{a,G}(\Omega). \end{cases}$$

Before stating our results, we introduce two notations $\mathscr{A}_\mu$ and $y_\epsilon(x)$, which are respectively defined by

$$\mathscr{A}_\mu \triangleq \inf_{u \in H^1_a(\Omega)\setminus\{0\}} \frac{\int_\Omega \left(|x|^{-2a}|\nabla u|^2 - \mu \frac{u^2}{|x|^{2(1+a)}} \right) dx}{\left(\int_\Omega |x|^{-bp}|u|^p dx \right)^{\frac{2}{p}}} \tag{2.3}$$

and

$$y_\epsilon(x) \triangleq \frac{C\epsilon^{\frac{1}{p-2}}}{|x|^{\sqrt{\bar{\mu}}-\sqrt{\bar{\mu}-\mu}} \left(\epsilon + |x|^{(p-2)\sqrt{\bar{\mu}-\mu}} \right)^{\frac{2}{p-2}}}, \tag{2.4}$$

where $\epsilon > 0$ and the constant $C = C(N, p, \mu) > 0$, depending only on N, p and μ. From [19], we know that $\mathscr{A}_\mu$ is independent of Ω and $y_\epsilon(x)$ satisfies the equations

$$\int_{\mathbb{R}^N} \left(|x|^{-2a}|\nabla y_\epsilon|^2 - \mu \frac{y_\epsilon^2}{|x|^{2(1+a)}} \right) dx = 1 \tag{2.5}$$

and

$$\int_{\mathbb{R}^N} |x|^{-bp} y_\epsilon^{p-1} \nu dx = \mathscr{A}_\mu^{-\frac{p}{2}} \int_{\mathbb{R}^N} \left(|x|^{-2a}\nabla y_\epsilon \nabla \nu - \mu \frac{y_\epsilon \nu}{|x|^{2(1+a)}} \right) dx$$

for all $\nu \in \mathscr{D}^{1,2}_a(\mathbb{R}^N)$, where $\mathscr{D}^{1,2}_a(\mathbb{R}^N)$ is the closure of $\mathcal{C}_0^\infty(\mathbb{R}^N)$ functions with respect to the norm $(\int_{\mathbb{R}^N} |x|^{-2a}|\nabla \cdot|^2 dx)^{1/2}$. In particular, we have (let $\nu = y_\epsilon$)

$$\int_{\mathbb{R}^N} |x|^{-bp} y_\epsilon^p dx = \mathscr{A}_\mu^{-\frac{p}{2}}. \tag{2.6}$$

We suppose that $Q(x)$ and $h(x, u)$ fulfill the following conditions.

(q.1) $Q \in \mathcal{C}(\overline{\Omega}) \cap L^\infty(\overline{\Omega})$, and $Q(x)$ is G-symmetric.
(q.2) $Q_+ \not\equiv 0$, where $Q_+ = \max\{0, Q\}$.
(h.1) $h \in \mathcal{C}(\overline{\Omega} \times \mathbb{R}, \mathbb{R})$, $h(x, t)$ is G-symmetric in $x \in \Omega$ for each $t \in \mathbb{R}$.
(h.2) There exist constants $C > 0$ and $q_0 \in (2, p)$ such that $0 \le h(x, t) \le C(1 + t^{q_0-1})$ for all $x \in \Omega$ and $t \ge 0$.
(h.3) $h(x, 0) = 0$ and $\lim_{t\to 0^+} h(x, t)/t = +\infty$ uniformly in $x \in \Omega$.
(h.4) There exists a constant $l > 0$ such that

$$\frac{1}{2} t h(x, t) - H(x, t) \ge -l|x|^{-bp}t^p$$

for all $x \in \Omega$ and $t \ge 0$, where $H(x, t) = \int_0^t h(x, s)ds$.

Since $0 \in \Omega$, we can choose $\varrho > 0$ small enough such that $B(0, 2\varrho) \subset \Omega$ and define a function $\phi \in \mathcal{C}^1_0(\Omega)$ such that $\phi(x) = 1$ on $B(0, \varrho)$, $\phi(x) = 0$ on $\Omega \setminus B(0, 2\varrho)$. Setting $V_\epsilon = \phi y_\epsilon / \|\phi y_\epsilon\|$, we obtain (see (3.10) for details)

$$\|V_\epsilon\| = 1 \quad \text{and} \quad \int_\Omega |x|^{-bp}|V_\epsilon|^p dx = \mathscr{A}_\mu^{-\frac{p}{2}} + O\left(\epsilon^{\frac{2}{p-2}}\right).$$

The main results of this paper are the following.

Theorem 2.1. *Suppose that* (q.1) *and* (q.2) *hold. If*

$$\int_{\Omega} Q(x)|x|^{-bp}|V_\epsilon|^p dx \geq \max\left\{Q_+(0),\, |G|^{\frac{2-p}{2}}(\mathscr{A}_0/\mathscr{A}_\mu)^{-\frac{p}{2}}\|Q_+\|_\infty\right\}\mathscr{A}_\mu^{-\frac{p}{2}} > 0 \tag{2.7}$$

for some $\epsilon > 0$, *then problem* $(\mathscr{P}_0^Q)$ *has at least one positive solution in* $H^1_{a,G}(\Omega)$.

Corollary 2.1. *Suppose that* (q.1) *and* (q.2) *hold. Then problem* $(\mathscr{P}_0^Q)$ *has at least one positive solution in* $H^1_{a,G}(\Omega)$ *if*

$$Q(0) > 0, \quad Q(0) \geq |G|^{\frac{2-p}{2}}(\mathscr{A}_0/\mathscr{A}_\mu)^{-\frac{p}{2}}\|Q_+\|_\infty \tag{2.8}$$

and $Q(x) \geq Q(0) + \gamma_0|x|^\vartheta$ *for some* $\gamma_0 > 0$, $\vartheta \in (0, 2\sqrt{\overline{\mu} - \mu})$ *and* $|x|$ *small.*

Theorem 2.2. *Suppose that* $Q_+(0) = 0$ *and* $|G| = +\infty$. *Then problem* $(\mathscr{P}_0^Q)$ *has infinitely many G-symmetric solutions.*

Theorem 2.3. *Let* $\overline{Q} > 0$ *be a constant. Suppose that* $Q(x) \equiv \overline{Q}$ *and* (h.1)–(h.4) *hold. Then there exists* $\sigma^* > 0$ *such that, for any* $\sigma \in (0, \sigma^*)$, *problem* $(\mathscr{P}_\sigma^{\overline{Q}})$ *possesses at least two positive solutions in* $H^1_{a,G}(\Omega)$.

Throughout this paper, we assume that $0 \in \Omega$ and $\Omega \subset \mathbb{R}^N (N \geq 3)$ is G-symmetric. Denote by $H^1_{a,G}(\Omega)$ the subspace of $H^1_a(\Omega)$ consisting of all G-symmetric functions. The dual space of $H^1_{a,G}(\Omega)$ ($H^1_a(\Omega)$, resp.) is denoted by $H^{-1}_{a,G}(\Omega)$ ($H^{-1}_a(\Omega)$, resp.). The ball of center x and radius r is denoted by $B(x, r)$. We employ $C, C_i (i = 1, 2, \ldots)$ to denote the positive constants, and denote by "$\rightarrow$" convergence in norm in a given Banach space X and by "$\rightharpoonup$" weak convergence. A functional $J \in \mathcal{C}^1(X, \mathbb{R})$ is said to satisfy the $(PS)_c$ condition if each sequence $\{u_n\}$ in X satisfying $J(u_n) \rightarrow c, J'(u_n) \rightarrow 0$ in X^* has a subsequence which strongly converges to some element in X. Hereafter, $L^r(\Omega, |x|^{-s})$ denotes the weighted $L^r(\Omega)$ space with the norm $(\int_\Omega |x|^{-s}|u|^r dx)^{1/r}$.

3. Existence and multiplicity results for problem $(\mathscr{P}_0^Q)$

We associate with problem $(\mathscr{P}_0^Q)$ a functional $\mathscr{F} : H^1_{a,G}(\Omega) \rightarrow \mathbb{R}$ given by

$$\mathscr{F}(u) = \frac{1}{2}\int_\Omega \left(|x|^{-2a}|\nabla u|^2 - \mu\frac{u^2}{|x|^{2(1+a)}}\right)dx - \frac{1}{p}\int_\Omega Q(x)|x|^{-bp}|u|^p dx. \tag{3.1}$$

By (q.1) and (2.1), we easily see that the functional $\mathscr{F} \in \mathcal{C}^1(H^1_{a,G}(\Omega), \mathbb{R})$. Now it is well known that there exists a one-to-one correspondence between the weak solutions of problem $(\mathscr{P}_0^Q)$ and the critical points of $\mathscr{F}$. More precisely, the weak solutions of $(\mathscr{P}_0^Q)$ are exactly the critical points of $\mathscr{F}$ by the following principle of symmetric criticality due to Palais (see Lemma 3.1), namely $u \in H^1_{a,G}(\Omega)$ satisfies $(\mathscr{P}_0^Q)$ if and only if

$$\int_\Omega \left(|x|^{-2a}\nabla u\nabla v - \mu\frac{uv}{|x|^{2(1+a)}}\right)dx - \int_\Omega Q(x)\frac{|u|^{p-2}uv}{|x|^{bp}}dx = 0, \quad \forall v \in H^1_a(\Omega). \tag{3.2}$$

Lemma 3.1. *Let* $Q(x)$ *be a G-symmetric function;* $\mathscr{F}'(u) = 0$ *in* $H^{-1}_{a,G}(\Omega)$ *implies* $\mathscr{F}'(u) = 0$ *in* $H^{-1}_a(\Omega)$.

Proof. Similar to the proof of [13, Lemma 1] (see also [12, Lemma 3.1]). □

Lemma 3.2. *Let* $\{u_n\}$ *be a weakly convergent sequence to* u *in* $H^1_{a,G}(\Omega)$ *such that* $|x|^{-2a}|\nabla u_n|^2 \rightharpoonup \eta$, $|x|^{-bp}|u_n|^p \rightharpoonup \nu$ *and* $|x|^{-2(1+a)}|u_n|^2 \rightharpoonup \widetilde{\nu}$ *in the sense of measures. Then there exists some at most countable set* $\mathscr{J}$, $\{\eta_j \geq 0\}_{j\in\mathscr{J}\cup\{0\}}$, $\{\nu_j \geq 0\}_{j\in\mathscr{J}\cup\{0\}}$, $\widetilde{\nu}_0 \geq 0$, $\{x_j\}_{j\in\mathscr{J}} \subset \overline{\Omega} \setminus \{0\}$ *such that*

(a) $\eta \geq |x|^{-2a}|\nabla u|^2 + \sum_{j\in\mathscr{J}} \eta_j\delta_{x_j} + \eta_0\delta_0$,
(b) $\nu = |x|^{-bp}|u|^p + \sum_{j\in\mathscr{J}} \nu_j\delta_{x_j} + \nu_0\delta_0$,
(c) $\widetilde{\nu} = |x|^{-2(1+a)}|u|^2 + \widetilde{\nu}_0\delta_0$,
(d) $\mathscr{A}_0\nu_j^{2/p} \leq \eta_j$,
(e) $\mathscr{A}_\mu\nu_0^{2/p} \leq \eta_0 - \mu\widetilde{\nu}_0$,

where δ_{x_j}, $j \in \mathscr{J} \cup \{0\}$, *is the Dirac-mass of* 1 *concentrated at* $x_j \in \overline{\Omega}$.

Proof. The proof is similar to that of the concentration-compactness principle in [20] and is omitted here. □

To find critical points of $\mathscr{F}$ we need the following local $(PS)_c$ condition, which is crucial for the proof of Theorem 2.1.

Lemma 3.3. *Suppose that* (q.1) *and* (q.2) *hold. Then the* $(PS)_c$ *condition in* $H^1_{a,G}(\Omega)$ *holds for* $\mathscr{F}(u)$ *if*

$$c < c_0^* \triangleq \frac{p-2}{2p} \min\left\{ \mathscr{A}_\mu^{\frac{p}{p-2}} Q_+(0)^{\frac{2}{2-p}}, |G| \mathscr{A}_0^{\frac{p}{p-2}} \|Q_+\|_\infty^{\frac{2}{2-p}} \right\}. \tag{3.3}$$

Proof. We follow the arguments of [13]. Let $\{u_n\}$ be a $(PS)_c$ sequence for $\mathscr{F}$ with c satisfying (3.3). It is easy to see that $\{u_n\}$ is bounded in $H^1_{a,G}(\Omega)$ and we may assume that $u_n \rightharpoonup u$ in $H^1_{a,G}(\Omega)$. By Lemma 3.2 there exist measures η, ν and $\widetilde{\nu}$ such that relations (a)–(e) of this lemma hold. Let $x_j \neq 0$ be a singular point of measures η and ν. As in paper [11], we define a function $\phi_\epsilon \in \mathcal{C}^1(\Omega)$ such that $\phi_\epsilon = 1$ in $B(x_j, \epsilon/2)$, $\phi_\epsilon = 0$ on $\Omega \setminus B(x_j, \epsilon)$ and $|\nabla \phi_\epsilon| \leq 4/\epsilon$. By Lemma 3.1, $\lim_{n\to\infty}\langle \mathscr{F}'(u_n), u_n\phi_\epsilon\rangle = 0$, hence, using (2.1) and the Hölder inequality, we obtain

$$\begin{aligned}
\int_\Omega \phi_\epsilon d\eta - \int_\Omega \mu\phi_\epsilon d\widetilde{\nu} - \int_\Omega Q(x)\phi_\epsilon d\nu &\leq \limsup_{n\to\infty} \int_\Omega |x|^{-2a}|u_n|\,|\nabla u_n|\,|\nabla\phi_\epsilon| dx \\
&\leq \sup_{n\geq 1}\left(\int_\Omega |x|^{-2a}|\nabla u_n|^2 dx\right)^{1/2} \limsup_{n\to\infty}\left(\int_\Omega |x|^{-2a}|u_n|^2|\nabla\phi_\epsilon|^2 dx\right)^{1/2} \\
&\leq C\left(\int_\Omega |x|^{-2a}|u|^2|\nabla\phi_\epsilon|^2 dx\right)^{1/2} \\
&\leq C\left(\int_{B(x_j,\epsilon)} |x|^{-2^*a}|u|^{2^*} dx\right)^{1/2^*} \left(\int_\Omega |\nabla\phi_\epsilon|^N dx\right)^{1/N} \\
&\leq C\left(\int_{B(x_j,\epsilon)} |x|^{-2a}|\nabla u|^2 dx\right)^{1/2}.
\end{aligned} \tag{3.4}$$

Letting $\epsilon \to 0$ in (3.4), we deduce from Lemma 3.2 that

$$Q(x_j)\nu_j \geq \eta_j. \tag{3.5}$$

This inequality says that the concentration of the measure ν cannot occur at points where $Q(x_j) \leq 0$, that is, if $Q(x_j) \leq 0$ then $\eta_j = \nu_j = 0$. Combining (3.5) and (d) of Lemma 3.2 we obtain that either (i) $\nu_j = 0$ or (ii) $\nu_j \geq (\mathscr{A}_0/\|Q_+\|_\infty)^{\frac{p}{p-2}}$. For the point $x = 0$, similarly as in the case $x_j \neq 0$, we get $\eta_0 - \mu\widetilde{\nu}_0 - Q(0)\nu_0 \leq 0$. This, combined with (e) of Lemma 3.2, implies that either (iii) $\nu_0 = 0$ or (iv) $\nu_0 \geq (\mathscr{A}_\mu/Q_+(0))^{\frac{p}{p-2}}$. We now show that (ii) and (iv) cannot occur. For every continuous nonnegative function ψ such that $0 \leq \psi(x) \leq 1$ on Ω, we deduce from (3.1) and (3.2) that

$$\begin{aligned}
c &= \lim_{n\to\infty}\left(\mathscr{F}(u_n) - \frac{1}{p}\langle\mathscr{F}'(u_n), u_n\rangle\right) \\
&= \left(\frac{1}{2} - \frac{1}{p}\right)\lim_{n\to\infty}\int_\Omega\left(|x|^{-2a}|\nabla u_n|^2 - \mu\frac{u_n^2}{|x|^{2(1+a)}}\right)dx \\
&\geq \frac{p-2}{2p}\limsup_{n\to\infty}\int_\Omega\left(|x|^{-2a}|\nabla u_n|^2 - \mu\frac{u_n^2}{|x|^{2(1+a)}}\right)\psi(x)dx.
\end{aligned}$$

If (ii) occurs, then the set $\mathscr{J}$ must be finite because the measure ν is bounded. Since functions u_n are G-symmetric, the measure ν must be G-invariant. This means that if $x_j \neq 0$ is a singular point of ν, so is gx_j for each $g \in G$, and the mass of ν concentrated at gx_j is the same for each $g \in G$. If we assume the existence of $j \in \mathscr{J}$ with $x_j \neq 0$ such that (ii) holds, then we choose ψ with compact support so that $\psi(gx_j) = 1$ for each $g \in G$ and we get

$$c \geq \frac{p-2}{2p}|G|\eta_j \geq \frac{p-2}{2p}|G|\mathscr{A}_0\nu_j^{2/p} \geq \frac{p-2}{2p}|G|\mathscr{A}_0^{\frac{p}{p-2}}\|Q_+\|_\infty^{\frac{2}{2-p}},$$

which contradicts (3.3). Similarly, if (iv) holds for $x = 0$, we choose ψ with compact support, so that $\psi(0) = 1$, and we obtain

$$c \geq \frac{p-2}{2p}(\eta_0 - \mu\widetilde{\nu}_0) \geq \frac{p-2}{2p}\mathscr{A}_\mu\nu_0^{2/p} \geq \frac{p-2}{2p}\mathscr{A}_\mu^{\frac{p}{p-2}}Q_+(0)^{\frac{2}{2-p}},$$

a contradiction with (3.3). Hence $\nu_j = 0$ for all $j \in \mathscr{J} \cup \{0\}$, and consequently we have $u_n \to u$ in $L^p(\Omega, |x|^{-bp})$. Finally, observe that $\mathscr{F}'(u) = 0$ and thus, by $\lim_{n\to\infty}\langle \mathscr{F}'(u_n) - \mathscr{F}'(u), u_n - u\rangle = 0$, we obtain $u_n \to u$ in $H_a^1(\Omega)$. □

As an immediate consequence of Lemma 3.3 we get the following result.

Corollary 3.1. *If $Q_+(0) = 0$ and $|G| = +\infty$, then the functional $\mathscr{F}$ satisfies the $(PS)_c$ condition for every $c \in \mathbb{R}$.*

Proof of Theorem 2.1. First, we choose $\epsilon > 0$ such that the condition (2.7) holds, where $V_\epsilon = \phi y_\epsilon / \|\phi y_\epsilon\|$ satisfies (3.8)–(3.10). It is trivial to check that there exist constants $\alpha_0 > 0$ and $\rho > 0$ such that $\mathscr{F}(u) \ge \alpha_0$ for all $\|u\| = \rho$. A simple calculation shows that there exists $\bar{t} > 0$ such that

$$\max_{t\ge 0} \mathscr{F}(tV_\epsilon) = \mathscr{F}(\bar{t}V_\epsilon) = \frac{p-2}{2p}\left\{\frac{\int_\Omega \left(|x|^{-2a}|\nabla V_\epsilon|^2 - \mu\frac{|V_\epsilon|^2}{|x|^{2(1+a)}}\right)dx}{\left(\int_\Omega Q(x)|x|^{-bp}|V_\epsilon|^p dx\right)^{\frac{2}{p}}}\right\}^{\frac{p}{p-2}}. \tag{3.6}$$

We now choose $t_0 > 0$ such that $\mathscr{F}(t_0V_\epsilon) < 0$ and $\|t_0V_\epsilon\| > \rho$ and set

$$c_0 = \inf_{\gamma\in\Gamma}\max_{t\in[0,1]} \mathscr{F}(\gamma(t)), \tag{3.7}$$

where $\Gamma = \{\gamma \in \mathcal{C}([0,1], H_{a,G}^1(\Omega)); \gamma(0) = 0, \mathscr{F}(\gamma(1)) < 0, \|\gamma(1)\| > \rho\}$. From (2.7), (3.6) and (3.7) and the definition of c_0^*, we deduce that

$$\begin{aligned} c_0 \le \mathscr{F}(\bar{t}V_\epsilon) &= \frac{p-2}{2p}\left\{\frac{\int_\Omega \left(|x|^{-2a}|\nabla V_\epsilon|^2 - \mu\frac{|V_\epsilon|^2}{|x|^{2(1+a)}}\right)dx}{\left(\int_\Omega Q(x)|x|^{-bp}|V_\epsilon|^p dx\right)^{\frac{2}{p}}}\right\}^{\frac{p}{p-2}} \\ &\le \frac{p-2}{2p}\left\{\frac{\int_\Omega \left(|x|^{-2a}|\nabla V_\epsilon|^2 - \mu\frac{|V_\epsilon|^2}{|x|^{2(1+a)}}\right)dx}{\left(\max\left\{Q_+(0), |G|^{\frac{2-p}{2}}(\mathscr{A}_0/\mathscr{A}_\mu)^{-\frac{p}{2}}\|Q_+\|_\infty\right\}\mathscr{A}_\mu^{-\frac{p}{2}}\right)^{\frac{2}{p}}}\right\}^{\frac{p}{p-2}} \\ &= \frac{p-2}{2p}\min\left\{\mathscr{A}_\mu^{\frac{p}{p-2}}Q_+(0)^{\frac{2}{2-p}}, |G|\mathscr{A}_0^{\frac{p}{p-2}}\|Q_+\|_\infty^{\frac{2}{2-p}}\right\} = c_0^*. \end{aligned}$$

If $c_0 < c_0^*$, then by Lemma 3.3, the $(PS)_c$ condition holds and the conclusion follows from the mountain pass theorem (see [21]). If $c_0 = c_0^*$, then $\gamma(t) = tt_0V_\epsilon$, with $0 \le t \le 1$, is a path in Γ such that $\max_{t\in[0,1]}\mathscr{F}(\gamma(t)) = c_0$. Consequently, either $\mathscr{F}'(\bar{t}V_\epsilon) = 0$ and we are done, or γ can be deformed to a path $\widetilde{\gamma} \in \Gamma$ with $\max_{t\in[0,1]}\mathscr{F}(\widetilde{\gamma}(t)) < c_0$ and we get a contradiction. This part of the proof shows that a nontrivial solution $u_0 \in H_{a,G}^1(\Omega)$ of problem $(\mathscr{P}_0^Q)$ exists. We now show that the solution u_0 can be chosen to be positive on Ω. Since $\mathscr{F}(u_0) = \mathscr{F}(|u_0|)$ and

$$0 = \langle \mathscr{F}'(u_0), u_0\rangle = \int_\Omega \left(|x|^{-2a}|\nabla u_0|^2 - \mu\frac{u_0^2}{|x|^{2(1+a)}}\right)dx - \int_\Omega Q(x)\frac{|u_0|^p}{|x|^{bp}}dx,$$

we have $\int_\Omega Q(x)|x|^{-bp}|u_0|^p dx > 0$. This implies $c_0 = \mathscr{F}(|u_0|) = \max_{t\ge 0}\mathscr{F}(t|u_0|)$. Thus, either $|u_0|$ is a critical point of $\mathscr{F}$ or $\gamma(t) = tt_0|u_0|$, with $\mathscr{F}(t_0|u_0|) < 0$, can be deformed, as above of the proof, to a path $\widetilde{\gamma}(t)$ with $\max_{t\in[0,1]}\mathscr{F}(\widetilde{\gamma}(t)) < c_0$, which is impossible. Therefore, we may assume that u_0 is nonnegative on Ω and the fact that $u_0 > 0$ on Ω follows by the strong maximum principle. □

Proof of Corollary 2.1. Let $y_\epsilon(x)$ be the extremal function satisfying (2.4)–(2.6). Choose $\phi \in \mathcal{C}_0^1(\Omega)$ so that $\phi \ge 0$ on Ω and $\phi(x) = 1$ on $B(0, \varrho)$, with $\varrho > 0$ to be determined. Using the methods in [3], we deduce from (2.4)–(2.6) that

$$\|\phi y_\epsilon\|^2 = \int_\Omega \left(|x|^{-2a}|\nabla(\phi y_\epsilon)|^2 - \mu|x|^{-2(1+a)}|\phi y_\epsilon|^2\right)dx = 1 + O\left(\epsilon^{\frac{2}{p-2}}\right), \tag{3.8}$$

$$\int_\Omega |x|^{-bp}|\phi y_\epsilon|^p dx = \mathscr{A}_\mu^{-\frac{p}{2}} + O\left(\epsilon^{\frac{p}{p-2}}\right). \tag{3.9}$$

Set $V_\epsilon = \phi y_\epsilon / \|\phi y_\epsilon\|$; then by (3.8) and (3.9) we get

$$\int_\Omega |x|^{-bp}|V_\epsilon|^p dx = \int_\Omega \frac{|x|^{-bp}|\phi y_\epsilon|^p}{\|\phi y_\epsilon\|^p} dx = \mathscr{A}_\mu^{-\frac{p}{2}} + O\left(\epsilon^{\frac{2}{p-2}}\right). \tag{3.10}$$

Let us now choose $\varrho > 0$ so that $Q(x) \geq Q(0) + \gamma_0 |x|^\vartheta$ for $|x| \leq \varrho$. Then we obtain from (3.10) that

$$\int_\Omega Q(x)|x|^{-bp}|V_\epsilon|^p dx = \int_\Omega (Q(x) - Q(0))|x|^{-bp}|V_\epsilon|^p dx + Q(0)\mathscr{A}_\mu^{-\frac{p}{2}} + O\left(\epsilon^{\frac{2}{p-2}}\right).$$

It is sufficient to show that

$$\int_\Omega (Q(x) - Q(0))|x|^{-bp}|V_\epsilon|^p dx + O\left(\epsilon^{\frac{2}{p-2}}\right) \geq 0 \tag{3.11}$$

for sufficiently small $\epsilon > 0$. We have

$$\begin{aligned}\int_\Omega (Q(x) - Q(0))|x|^{-bp}|V_\epsilon|^p dx &= \int_{|x|\leq\varrho} (Q(x) - Q(0))|x|^{-bp}|V_\epsilon|^p dx + \int_{|x|\geq\varrho} (Q(x) - Q(0))|x|^{-bp}|V_\epsilon|^p dx \\ &\geq \gamma_0 \int_{|x|\leq\varrho} \frac{|x|^{\vartheta-bp}|y_\epsilon|^p}{\|\phi y_\epsilon\|^p} dx + \int_{|x|\geq\varrho} \frac{(Q(x) - Q(0))|\phi y_\epsilon|^p}{|x|^{bp}\|\phi y_\epsilon\|^p} dx = I_1 + I_2.\end{aligned}$$

For $\epsilon > 0$ sufficiently small, we deduce from (2.4)–(2.6), (3.8) and the fact $N - 1 + \vartheta - (b + \sqrt{\bar\mu} - \sqrt{\bar\mu - \mu})p > -1$, $N - 1 + \vartheta - (b + \sqrt{\bar\mu} - \sqrt{\bar\mu - \mu})p - (p-2)\sqrt{\bar\mu - \mu} \cdot \frac{2p}{p-2} < -1$ that

$$\begin{aligned} I_1 &= \gamma_0 \int_{|x|\leq\varrho} \frac{|x|^{\vartheta-bp}|y_\epsilon|^p}{\|\phi y_\epsilon\|^p} dx \\ &= \frac{\gamma_0}{\left(1 + O\left(\epsilon^{\frac{2}{p-2}}\right)\right)^{\frac{p}{2}}} \int_{|x|\leq\varrho} |x|^{\vartheta-bp} \left[C\epsilon^{\frac{1}{p-2}} |x|^{-\sqrt{\bar\mu}+\sqrt{\bar\mu-\mu}} \left(\epsilon + |x|^{(p-2)\sqrt{\bar\mu-\mu}}\right)^{\frac{2}{2-p}} \right]^p dx \\ &= \frac{\gamma_0 C^p \epsilon^{\frac{p}{p-2}}}{\left(1 + O\left(\epsilon^{\frac{2}{p-2}}\right)\right)^{\frac{p}{2}}} \int_{|x|\leq\varrho} |x|^{\vartheta-(b+\sqrt{\bar\mu}-\sqrt{\bar\mu-\mu})p} \left(\epsilon + |x|^{(p-2)\sqrt{\bar\mu-\mu}}\right)^{\frac{2p}{2-p}} dx \\ &= \frac{C\epsilon^{\frac{\vartheta}{(p-2)\sqrt{\bar\mu-\mu}}}}{\left(1 + O\left(\epsilon^{\frac{2}{p-2}}\right)\right)^{\frac{p}{2}}} \int_{|x|\leq\varrho} \frac{\left(\frac{|x|}{\epsilon^{\frac{1}{(p-2)\sqrt{\bar\mu-\mu}}}}\right)^{\vartheta-(b+\sqrt{\bar\mu}-\sqrt{\bar\mu-\mu})p}}{\left[1 + \left(\frac{|x|}{\epsilon^{\frac{1}{(p-2)\sqrt{\bar\mu-\mu}}}}\right)^{(p-2)\sqrt{\bar\mu-\mu}}\right]^{\frac{2p}{p-2}}} d\left(\frac{x}{\epsilon^{\frac{1}{(p-2)\sqrt{\bar\mu-\mu}}}}\right) \\ &\geq C\epsilon^{\frac{\vartheta}{(p-2)\sqrt{\bar\mu-\mu}}} \left\{ \left(\int_0^1 + \int_1^{\varrho\epsilon^{\frac{-1}{(p-2)\sqrt{\bar\mu-\mu}}}} \right) \frac{r^{N-1+\vartheta-(b+\sqrt{\bar\mu}-\sqrt{\bar\mu-\mu})p}}{\left(1 + r^{(p-2)\sqrt{\bar\mu-\mu}}\right)^{\frac{2p}{p-2}}} dr \right\} \\ &\geq \overline{C}_1 \epsilon^{\frac{\vartheta}{(p-2)\sqrt{\bar\mu-\mu}}}, \quad \vartheta \in \left(0, 2\sqrt{\bar\mu - \mu}\right) \end{aligned}$$

and

$$\begin{aligned} |I_2| &\leq \int_{|x|\geq\varrho} \frac{|Q(x) - Q(0)|\,|\phi y_\epsilon|^p}{|x|^{bp}\|\phi y_\epsilon\|^p} dx \\ &\leq C \int_{|x|\geq\varrho} \frac{\epsilon^{\frac{p}{p-2}}}{|x|^{(b+\sqrt{\bar\mu}-\sqrt{\bar\mu-\mu})p}\left(\epsilon + |x|^{(p-2)\sqrt{\bar\mu-\mu}}\right)^{\frac{2p}{p-2}}} dx \\ &= C \int_{|x|\geq\varrho} \frac{\left(\frac{|x|}{\epsilon^{\frac{1}{(p-2)\sqrt{\bar\mu-\mu}}}}\right)^{-(b+\sqrt{\bar\mu}-\sqrt{\bar\mu-\mu})p}}{\left[1 + \left(\frac{|x|}{\epsilon^{\frac{1}{(p-2)\sqrt{\bar\mu-\mu}}}}\right)^{(p-2)\sqrt{\bar\mu-\mu}}\right]^{\frac{2p}{p-2}}} d\left(\frac{x}{\epsilon^{\frac{1}{(p-2)\sqrt{\bar\mu-\mu}}}}\right) \end{aligned}$$

$$\leq C\int_{\varrho\epsilon^{\frac{-1}{(p-2)\sqrt{\bar\mu-\mu}}}}^{+\infty}\frac{r^{N-1-(b+\sqrt{\bar\mu}-\sqrt{\bar\mu-\mu})p}}{\left(1+r^{(p-2)\sqrt{\bar\mu-\mu}}\right)^{\frac{2p}{p-2}}}dr\leq \overline{C}_2\epsilon^{\frac{p}{p-2}},$$

where $\overline{C}_1>0$ and $\overline{C}_2>0$ are constants independent of ϵ. Since $0<\frac{\vartheta}{(p-2)\sqrt{\bar\mu-\mu}}<\frac{2}{p-2}<\frac{p}{p-2}$, inequality (3.11) follows as $\epsilon>0$ small enough. Therefore we conclude from (2.8), (3.10) and (3.11) that

$$\begin{aligned}\int_\Omega Q(x)|x|^{-bp}|V_\epsilon|^p dx&=\int_\Omega (Q(x)-Q(0))|x|^{-bp}|V_\epsilon|^p dx+Q(0)\mathscr{A}_\mu^{-\frac{p}{2}}+O\left(\epsilon^{\frac{2}{p-2}}\right)\\&\geq Q(0)\mathscr{A}_\mu^{-\frac{p}{2}}\geq \max\left\{Q_+(0),\,|G|^{\frac{2-p}{2}}(\mathscr{A}_0/\mathscr{A}_\mu)^{-\frac{p}{2}}\|Q_+\|_\infty\right\}\mathscr{A}_\mu^{-\frac{p}{2}}>0.\end{aligned}$$

By Theorem 2.1 and the above inequality, we obtain the conclusion. □

To prove Theorem 2.2 we need the following version of symmetric mountain pass theorem (cf. [22, Theorem 9.12]).

Lemma 3.4. *Let E be an infinite dimensional Banach space and let $\mathscr{F}\in\mathcal{C}^1(E,\mathbb{R})$ be an even functional satisfying the $(PS)_c$ condition for each c and $\mathscr{F}(0)=0$. Further, we suppose that:*

(i) *there exist constants $\overline{\alpha}>0$ and $\rho>0$ such that $\mathscr{F}(u)\geq\overline{\alpha}$ for all $\|u\|=\rho$;*
(ii) *there exist an increasing sequence of subspaces $\{E_m\}$ of E, with $\dim E_m=m$, such that for every m one can find a constant $R_m>0$ such that $\mathscr{F}(u)\leq 0$ for all $u\in E_m$ with $\|u\|\geq R_m$.*

Then $\mathscr{F}$ possesses a sequence of critical values $\{c_m\}$ tending to ∞ as $m\to\infty$.

Proof of Theorem 2.2. Applying Lemma 3.4 with $E=H^1_{a,G}(\Omega)$, we deduce from (2.3) and (3.1) that

$$\begin{aligned}\mathscr{F}(u)&=\frac12\int_\Omega\left(|x|^{-2a}|\nabla u|^2-\mu\frac{u^2}{|x|^{2(1+a)}}\right)dx-\frac1p\int_\Omega Q(x)|x|^{-bp}|u|^p dx\\&\geq\frac12\|u\|^2-\frac1p\|Q\|_\infty\mathscr{A}_\mu^{-\frac{p}{2}}\|u\|^p.\end{aligned}$$

Since $p>2$, there exists $\overline{\alpha}>0$ and $\rho>0$ such that $\mathscr{F}(u)\geq\overline{\alpha}$ for all u with $\|u\|=\rho$. To find a suitable sequence of finite dimensional subspaces of $H^1_{a,G}(\Omega)$, we set $\omega=\{x\in\Omega;\,Q(x)>0\}$. Since the set ω is G-symmetric, we can define $H^1_{a,G}(\omega)$, which is the subspace of G-symmetric functions of $H^1_a(\omega)$. By extending functions in $H^1_{a,G}(\omega)$ to 0 outside ω we can assume that $H^1_{a,G}(\omega)\subset H^1_{a,G}(\Omega)$. Let $\{E_m\}$ be an increasing sequence of subspaces of $H^1_{a,G}(\omega)$ with $\dim E_m=m$ for each m. Then there exists a constant $\epsilon(m)>0$ such that

$$\int_\omega Q(x)|x|^{-bp}|\nu|^p dx\geq\epsilon(m)\quad\text{for all }\nu\in E_m,\ \text{with }\|\nu\|=1.$$

Consequently, if $0\neq u\in E_m$ then we write $u=t\nu$, with $t=\|u\|$ and $\|\nu\|=1$. Thus we have

$$\mathscr{F}(u)=\frac12 t^2-\frac1p t^p\int_\omega Q(x)|x|^{-bp}|\nu|^p dx\leq\frac12 t^2-\frac1p\epsilon(m)t^p\leq 0$$

for t large enough. By Lemma 3.4 and Corollary 3.1 we conclude that there exists a sequence of critical values $c_m\to\infty$ as $m\to\infty$ and the results follow. □

4. Multiplicity results for problem $(\mathscr{P}_\sigma^{\overline{Q}})$

The aim of this section is to discuss problem $(\mathscr{P}_\sigma^{\overline{Q}})$ and prove Theorem 2.3; here $Q(x)\equiv\overline{Q}>0$ is a constant. Since we are interested in positive G-symmetric solutions of $(\mathscr{P}_\sigma^{\overline{Q}})$, we assume that $h(x,t)=0$ for $x\in\Omega$ and $t\leq 0$ without changing the symbol. The energy functional corresponding to problem $(\mathscr{P}_\sigma^{\overline{Q}})$ is defined by

$$J_\sigma(u)=\frac12\int_\Omega\left(|x|^{-2a}|\nabla u|^2-\mu\frac{u^2}{|x|^{2(1+a)}}\right)dx-\frac{\overline{Q}}{p}\int_\Omega\frac{|u^+|^p}{|x|^{bp}}dx-\sigma\int_\Omega H(x,u)dx,\tag{4.1}$$

where $u^+=\max\{0,u\}$ and $H(x,u)=\int_0^u h(x,t)dt$. By (2.1) and (h.1)–(h.4), we easily see that $J_\sigma\in\mathcal{C}^1(H^1_{a,G}(\Omega),\mathbb{R})$ and there exists a one-to-one correspondence between the weak solutions of $(\mathscr{P}_\sigma^{\overline{Q}})$ and the critical points of J_σ. Moreover, an analogously symmetric criticality principle of Lemma 3.1 clearly holds; hence the weak solutions of $(\mathscr{P}_\sigma^{\overline{Q}})$ are exactly the critical points of the functional J_σ.

Lemma 4.1. *Suppose that* (h.1)–(h.4) *hold. Then there exists $\sigma_1^* > 0$ such that the functional J_σ satisfies the $(PS)_c$ condition for every $\sigma \in (0, \sigma_1^*)$ if*

$$c < \frac{p-2}{2p}\mathscr{A}_\mu^{\frac{p}{p-2}}\overline{Q}^{\frac{2}{2-p}}. \tag{4.2}$$

Proof. Let $\{u_n\} \subset H^1_{a,G}(\Omega)$ be a $(PS)_c$ sequence for J_σ with c satisfying (4.2). By (h.1)–(h.4), we easily check that $\psi(x,t) \triangleq \overline{Q}|x|^{-bp}(t^+)^{p-1} + \sigma h(x,t)$ satisfies the Ambrosetti–Rabinowitz condition

$$0 \le \xi\Psi(x,t) \le t\psi(x,t) \quad \text{for all } |t| \ge T, \tag{4.3}$$

and for some $\xi \in (2,p)$, where $\Psi(x,t) = \int_0^t \psi(x,s)ds$. Therefore we deduce from (4.2) and (4.3) that the sequence $\{u_n\}$ is bounded in $H^1_{a,G}(\Omega)$. Consequently, just as in Lemma 3.3, we may assume that $u_n \rightharpoonup u$ in $H^1_{a,G}(\Omega)$ and $u_n \to u$ a.e. in Ω. Moreover, in view of (h.2), we may also assume

$$\int_\Omega u_n h(x,u_n)dx \to \int_\Omega uh(x,u)dx, \qquad \int_\Omega H(x,u_n)dx \to \int_\Omega H(x,u)dx \tag{4.4}$$

as $n \to \infty$. By (4.4) and the standard argument, we easily see that u is a critical point of J_σ. Hence, we deduce from (h.4) that

$$\begin{aligned} J_\sigma(u) &= \frac{p-2}{2p}\overline{Q}\int_\Omega |x|^{-bp}|u^+|^p dx + \sigma\int_\Omega\left(\frac{1}{2}uh(x,u) - H(x,u)\right)dx \\ &= \frac{p-2}{2p}\overline{Q}\int_{\Omega_+} |x|^{-bp}u^p dx + \sigma\int_{\Omega_+}\left(\frac{1}{2}uh(x,u) - H(x,u)\right)dx \\ &\ge \left(\frac{p-2}{2p}\overline{Q} - \sigma l\right)\int_{\Omega_+}|x|^{-bp}u^p dx, \end{aligned} \tag{4.5}$$

where $\Omega_+ = \{x \in \Omega; u(x) > 0\}$. Setting $\sigma_1^* = \frac{p-2}{2pl}\overline{Q}$, we obtain from (4.5) that $J_\sigma(u) \ge 0$ for every $\sigma \in (0, \sigma_1^*)$.

Now we set $v_n = u_n - u$, then apply the Brezis–Lieb Lemma [23] to the sequence $|x|^{-bp}|u_n^+|^p$ and use the conditions (h.1)–(h.4) and the fact that u is a critical point of J_σ to obtain

$$\|v_n\|^2 = \overline{Q}\int_\Omega |x|^{-bp}|v_n^+|^p + o(1) \tag{4.6}$$

and

$$J_\sigma(u) + \frac{1}{2}\|v_n\|^2 - \frac{1}{p}\overline{Q}\int_\Omega |x|^{-bp}|v_n^+|^p = c + o(1). \tag{4.7}$$

Hence, for a subsequence of $\{v_n\}$ one gets

$$\|v_n\|^2 \to k \ge 0 \quad \text{and} \quad \overline{Q}\int_\Omega |x|^{-bp}|v_n^+|^p dx \to k \quad \text{as } n \to \infty.$$

It follows from (2.3) that $k \ge \mathscr{A}_\mu(k/\overline{Q})^{2/p}$, which implies either $k = 0$ or $k \ge \mathscr{A}_\mu^{\frac{p}{p-2}}\overline{Q}^{\frac{2}{2-p}}$. If $k \ge \mathscr{A}_\mu^{\frac{p}{p-2}}\overline{Q}^{\frac{2}{2-p}}$, we deduce from (4.5)–(4.7) and the fact $J_\sigma(u) \ge 0$ that

$$c = J_\sigma(u) + \frac{p-2}{2p}k \ge \frac{p-2}{2p}\mathscr{A}_\mu^{\frac{p}{p-2}}\overline{Q}^{\frac{2}{2-p}},$$

which contradicts (4.2). Therefore, we get $\|v_n\|^2 \to 0$ as $n \to \infty$, and hence, $u_n \to u$ in $H^1_{a,G}(\Omega)$. The proof of this lemma is completed. □

Lemma 4.2. *Suppose that* (h.1)–(h.4) *hold. Then there exists $\sigma_2^* > 0$ such that for any $\sigma \in (0, \sigma_2^*)$ the following geometric conditions for $J_\sigma(u)$ hold:*

(i) *$J_\sigma(0) = 0$, there exist $\widetilde{\alpha} > 0$, $\rho > 0$ such that $J_\sigma(u) \ge \widetilde{\alpha}$ for all $\|u\| = \rho$;*
(ii) *there exists $e \in H^1_{a,G}(\Omega)$ such that $\|e\| > \rho$ and $J_\sigma(e) \le 0$.*

Proof. According to (h.1)–(h.4), the functional $\mathscr{H} : H^1_{a,G}(\Omega) \to \mathbb{R}$, given by $\mathscr{H}(u) \triangleq \int_\Omega H(x,u)dx$ is continuous. Hence, given $\epsilon > 0$, there exists $\rho_1 > 0$ such that $|\mathscr{H}(u)| < \epsilon$ if $\|u\| \le \rho_1$. Then we obtain from (2.3) that

$$\begin{aligned} J_\sigma(u) &= \frac{1}{2}\|u\|^2 - \frac{1}{p}\overline{Q}\int_\Omega |x|^{-bp}|u^+|^p dx - \sigma\int_\Omega H(x,u)dx \\ &\ge \left(\frac{1}{2} - \frac{1}{p}\overline{Q}\mathscr{A}_\mu^{-p/2}\|u\|^{p-2}\right)\|u\|^2 - \sigma\epsilon. \end{aligned}$$

Let $\rho \in (0,\rho_1)$ be such that $J_\sigma(u) \ge \frac{1}{4}\|u\|^2 - \sigma\epsilon$ if $\|u\| \le \rho$. Setting $\sigma_2^* = \rho^2/8\epsilon$, we obtain $J_\sigma(u) \ge \rho^2/8$ if $\|u\| = \rho$ for every $\sigma \in (0,\sigma_2^*)$, which implies (i). On the other hand, since $p > 2$ and $\mathscr{H}(u) \ge 0$, we easily see that there exists $\widetilde{u} \in H^1_{a,G}(\Omega)\setminus\{0\}$ such that $J_\sigma(t\widetilde{u}) \to -\infty$ as $t \to +\infty$. Thus (ii) follows. □

Lemma 4.3. *Suppose that* (h.1)–(h.4) *hold. Then there exists some $v_0 \in H^1_{a,G}(\Omega)$, $v_0 \ge 0$ and $v_0 \not\equiv 0$ on Ω, such that*

$$\sup_{t\ge 0} J_\sigma(tv_0) < \frac{p-2}{2p}\mathscr{A}_\mu^{\frac{p}{p-2}}\overline{Q}^{\frac{2}{2-p}}. \tag{4.8}$$

Proof. We follow the arguments of [5, Lemma 3]. Recall that $V_\epsilon = \phi y_\epsilon/\|\phi y_\epsilon\|$, which satisfies (3.8)–(3.10). In the following, we will show that V_ϵ satisfies (4.8) for $\epsilon > 0$ sufficiently small. Set

$$\Phi(t) = J_\sigma(tV_\epsilon) = \frac{1}{2}t^2 - \frac{\overline{Q}}{p}t^p\int_\Omega |x|^{-bp}|V_\epsilon|^p dx - \sigma\int_\Omega H(x,tV_\epsilon)dx$$

and

$$\widetilde{\Phi}(t) = J_\sigma(tV_\epsilon) = \frac{1}{2}t^2 - \frac{\overline{Q}}{p}t^p\int_\Omega |x|^{-bp}|V_\epsilon|^p dx$$

with $t \ge 0$; we can see easily that $\widetilde{\Phi}$ has a unique maximum in positive t at some $\widetilde{t}_\epsilon > 0$ at which $\widetilde{\Phi}'(t)$ becomes zero. By simple calculation and (3.10), we obtain

$$\begin{aligned} \sup_{t\ge 0}\widetilde{\Phi}(t) &= \widetilde{\Phi}(\widetilde{t}_\epsilon) = \frac{p-2}{2p}\left(\overline{Q}\int_\Omega |x|^{-bp}|V_\epsilon|^p dx\right)^{\frac{2}{2-p}} \\ &= \frac{p-2}{2p}\mathscr{A}_\mu^{\frac{p}{p-2}}\overline{Q}^{\frac{2}{2-p}} + O\left(\epsilon^{\frac{2}{p-2}}\right). \end{aligned} \tag{4.9}$$

Using the definition of $\Phi(t)$, we get $\Phi(t) \le \widetilde{\Phi}(t)$ for all $t \ge 0$. This, combined with (4.9) and the fact $\widetilde{\Phi}(t)$ is increasing on the interval $[0,\widetilde{t}_\epsilon]$, implies that there exists $T_0 \in (0,\widetilde{t}_\epsilon)$ satisfying

$$\sup_{0\le t\le T_0}\Phi(t) \le \sup_{0\le t\le T_0}\widetilde{\Phi}(t) < \frac{p-2}{2p}\mathscr{A}_\mu^{\frac{p}{p-2}}\overline{Q}^{\frac{2}{2-p}}. \tag{4.10}$$

Since $H(x,t)$ is nondecreasing with respect to the second variable for $h(x,t) \ge 0$, we conclude from (4.9) and the fact $0 \in \Omega$ that there exists a constant $\rho_0 \in (0,\varrho)$ which is very small, such that

$$\begin{aligned} \sup_{t\ge T_0}\Phi(t) &< \sup_{t\ge 0}\widetilde{\Phi}(t) - \sigma\int_\Omega H(x,T_0V_\epsilon)dx \\ &\le \frac{p-2}{2p}\mathscr{A}_\mu^{\frac{p}{p-2}}\overline{Q}^{\frac{2}{2-p}} + C\epsilon^{\frac{2}{p-2}} - \sigma\int_{B(0,\rho_0)} H(x,T_0V_\epsilon)dx. \end{aligned} \tag{4.11}$$

Let $0 < \epsilon < \rho_0^{(p-2)\sqrt{\bar{\mu}-\mu}}$. Then for $x \in B(0,\rho_0)$, we deduce from (2.4) and (3.8) that

$$\begin{aligned} V_\epsilon(x) &= \frac{\phi y_\epsilon}{\left(1+O\left(\epsilon^{\frac{2}{p-2}}\right)\right)^{\frac{1}{2}}} \ge \frac{C\epsilon^{\frac{1}{p-2}}}{|x|^{\sqrt{\bar{\mu}}-\sqrt{\bar{\mu}-\mu}}\left(\epsilon + |x|^{(p-2)\sqrt{\bar{\mu}-\mu}}\right)^{\frac{2}{p-2}}} \\ &\ge \frac{C\epsilon^{\frac{1}{p-2}}}{\rho_0^{\sqrt{\bar{\mu}}-\sqrt{\bar{\mu}-\mu}}\left(2\rho_0^{(p-2)\sqrt{\bar{\mu}-\mu}}\right)^{\frac{2}{p-2}}} \triangleq C(p,\mu,\rho_0)\epsilon^{\frac{1}{p-2}}. \end{aligned} \tag{4.12}$$

On the other hand, we obtain from (2.4) and (3.8) that

$$V_\epsilon = \frac{\phi y_\epsilon}{\|\phi y_\epsilon\|} \le \frac{C\epsilon^{\frac{1}{p-2}}}{|x|^{\sqrt{\bar\mu}-\sqrt{\bar\mu-\mu}}\left(\epsilon + |x|^{(p-2)\sqrt{\bar\mu-\mu}}\right)^{\frac{2}{p-2}}} \to 0 \quad \text{as } \epsilon \to 0.$$

This, combined with (4.11) and (4.12) and $\lim_{t\to 0^+} H(x,t)/t^2 = +\infty$, which is directly got from (h.3), implies that

$$\begin{aligned}
\sup_{t\ge T_0} \Phi(t) &< \sup_{t\ge 0} \widetilde{\Phi}(t) - \sigma \int_\Omega H(x, T_0 V_\epsilon)dx \\
&\le \frac{p-2}{2p}\mathscr{A}_\mu^{\frac{p}{p-2}}\overline{Q}^{\frac{2}{2-p}} + \epsilon^{\frac{2}{p-2}}\left(C - \sigma T_0^2 \int_{B(0,\rho_0)} \frac{H(x,T_0V_\epsilon)}{(T_0V_\epsilon)^2}\cdot\frac{V_\epsilon^2}{\epsilon^{\frac{2}{p-2}}}dx\right) \\
&\le \frac{p-2}{2p}\mathscr{A}_\mu^{\frac{p}{p-2}}\overline{Q}^{\frac{2}{2-p}} + \epsilon^{\frac{2}{p-2}}\left(C - \sigma T_0^2 C^2(p,\mu,\rho_0) \int_{B(0,\rho_0)} \frac{H(x,T_0V_\epsilon)}{(T_0V_\epsilon)^2}dx\right) \\
&< \frac{p-2}{2p}\mathscr{A}_\mu^{\frac{p}{p-2}}\overline{Q}^{\frac{2}{2-p}}
\end{aligned} \tag{4.13}$$

for $\epsilon > 0$ sufficiently small. Therefore we deduce (4.8) from (4.10) and (4.13) and the results follow. □

Proof of Theorem 2.3. Taking $\rho > 0$ and $\sigma^* = \min\{\sigma_1^*, \sigma_2^*\}$, for $0 < \sigma < \sigma^*$, given in the proofs of Lemmas 4.1 and 4.2, we define

$$\overline{B(0,\rho)} = \{u \in H^1_{a,G}(\Omega);\ \|u\| \le \rho\} \quad \text{and} \quad c_1 \triangleq \inf_{\overline{B(0,\rho)}} J_\sigma(u).$$

Since the metric space $\overline{B(0,\rho)}$ with the distance $d(u,v) = \|u - v\|$ is complete, we conclude from the Ekeland variational principle that there exists a sequence $\{u_n\} \subset \overline{B(0,\rho)}$ such that $J_\sigma(u) \to c_1$ and $J'_\sigma(u_n) \to 0$ as $n \to \infty$.

Let $\widetilde{u} \in H^1_{a,G}(\Omega) \cap L^\infty(\Omega)$ be such that $\widetilde{u} \ne 0$. Setting $M_1 = \|\widetilde{u}\|^2/(\sigma\int_\Omega |\widetilde{u}|^2dx)$, we deduce from (h.3) that there exists $T_1 > 0$ such that $H(x,t) \ge M_1 t^2$ for $0 < t < T_1$. Therefore, for every $\sigma \in (0,\sigma^*)$ and $0 < r < \min\{\rho, T_1/\|\widetilde{u}\|_\infty\}$, we obtain from (4.1) that

$$\begin{aligned}
J_\sigma(r\widetilde{u}) &= \frac{1}{2}r^2\|\widetilde{u}\|^2 - \frac{1}{p}\overline{Q}r^p \int_\Omega |x|^{-bp}|\widetilde{u}^+|^p dx - \sigma\int_\Omega H(x, r\widetilde{u})dx \\
&< \frac{1}{2}r^2\|\widetilde{u}\|^2 - \sigma\int_\Omega M_1|r\widetilde{u}|^2dx = -\frac{1}{2}r^2\|\widetilde{u}\|^2 < 0,
\end{aligned}$$

which implies $c_1 < 0 < \frac{p-2}{2p}\mathscr{A}_\mu^{\frac{p}{p-2}}\overline{Q}^{\frac{2}{2-p}}$. By Lemma 4.3, J_σ possesses a critical point u_1 with $J_\sigma(u_1) = c_1 < 0$. Taking $u_1^- = \min\{0, u_1\}$ as a test function, we get $0 = \langle J'_\sigma(u_1), u_1^-\rangle = \|u_1^-\|^2$, which implies $u_1 \ge 0$ in Ω. By the strong maximum principle and the symmetric criticality principle, we obtain that u_1 is a positive G-symmetric solution of problem $(\mathscr{P}_\sigma^{\overline{Q}})$.

On the other hand, we define

$$c_2 \triangleq \inf_{\gamma\in\Gamma}\max_{t\in[0,1]} J_\sigma(\gamma(t)),$$

where $\Gamma = \{\gamma \in \mathcal{C}([0,1], H^1_{a,G}(\Omega));\ \gamma(0) = 0,\ \gamma(1) = e\}$. It follows from Lemmas 4.2 and 4.3 that

$$0 < \widetilde{\alpha} \le c_2 < \frac{p-2}{2p}\mathscr{A}_\mu^{\frac{p}{p-2}}\overline{Q}^{\frac{2}{2-p}}.$$

If $\sigma \in (0,\sigma^*)$, then c_2 is a critical value of J_σ by the mountain pass theorem. Therefore, similar to the arguments above, problem $(\mathscr{P}_\sigma^{\overline{Q}})$ admits another positive G-symmetric solution u_2 with $J_\sigma(u_2) = c_2 > 0$. □

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